

# Some Polyomino Tilings of the Plane

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**Abstract.** We calculate the generating functions for the number of tilings of rectangles of various widths by the right tromino, the  $L$  tetromino, and the  $T$  tetromino. This allows us to place lower bounds on the entropy of tilings of the plane by each of these. For the  $T$  tetromino, we also derive a lower bound from the solution of the Ising model in two dimensions.

## 1 Introduction

Tilings of the plane are of interest both to statistical physicists and recreational mathematicians. While the number of ways to tile a lattice with dominoes can be calculated by expressing it as a determinant [1, 2], telling whether a finite collection of shapes can tile the plane at all is undecidable [3, 4].

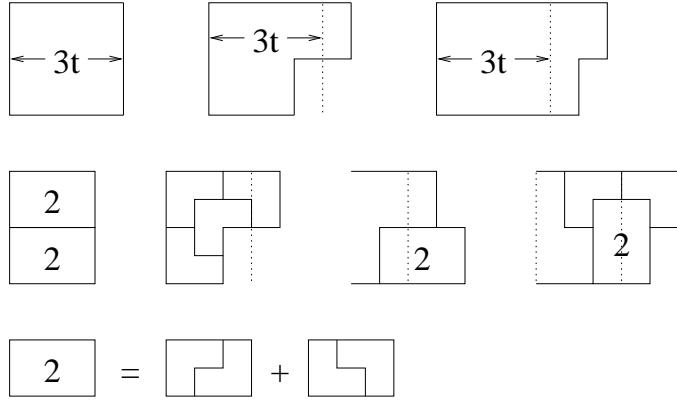
In the 1994 edition of his wonderful book *Polyominoes* [5], Solomon W. Golomb states that the problem of determining how many ways a  $4 \times n$  rectangle can be tiled by right trominoes “appears to be a challenging problem with reasonable hope of an attainable solution.” Here we solve this problem, and several more. Specifically, we find the generating functions for tilings of  $4 \times n$  and  $5 \times n$  rectangles by the right tromino, and tilings of  $4 \times n$  rectangles by the  $L$  and  $T$  tetrominoes.

The number of ways of tiling a  $m \times n$  rectangle with any finite collection of shapes, where  $m$  is fixed, can be found by calculating the  $n$ th power of a transfer matrix whose rows and columns correspond to the various interface shapes a partial tiling can have. As  $n$  grows large, the number of tilings  $N(n)$  grows as  $\lambda^n$  where  $\lambda$  is the largest eigenvalue of this matrix. In general, the number of interface shapes and therefore the size of the transfer matrix grows exponentially in  $m$ . However, if a small number of interface shapes suffice to generate all possible tilings, then we can use a small transfer matrix, and  $\lambda$  will then be an algebraic number of low degree. In fact, we find that for the problems considered here the largest matrix we need is  $4 \times 4$ .

By calculating  $\lambda$  for these rectangles, we can place a lower bound on the entropy per site of tilings of the plane by these polyominoes. In the case of the  $T$  tetromino, we improve this bound by using the exact solution of the Ising model in two dimensions.

## 2 The right tromino

Consider tilings of a  $4 \times n$  rectangle with right trominoes, where  $n$  is a multiple of 3. The number of interfaces seems potentially large. However, it turns out that we only need to think about three, so that a  $3 \times 3$  matrix will suffice. These are shown in Figure 1, with the various kinds of transitions that can take place as the tiling grows to the right.



**Fig. 1.** The three kinds of interfaces for the right tromino on rectangles of width 4, straight, deep jog, and shallow jog, and the transitions between them.

For these three interfaces, which we call ‘straight,’ ‘deep jog,’ and ‘shallow jog,’ we define  $N(t)$ ,  $N_1(t)$  and  $N_2(t)$  respectively as the number of tilings when there are  $n = 3t$  columns to the left of the dotted line. Obviously  $N_1(t)$  and  $N_2(t)$  stay the same if we count tilings of their vertical reflections instead. Our goal is to find  $N(t)$ . To express these, we write them as generating functions,

$$G(z) = \sum_t N(t) z^t$$

and similarly for  $G_1(z)$  and  $G_2(z)$ .

From Figure 1, we see that a straight interface can make a transition to itself in four ways, each of which increases  $t$  by 1. It has one transition each to a deep jog and its reflection, each of which has one transition back to a straight, and both of these increase  $t$  by 1. Deep and shallow jogs of a given orientation have two transitions in each direction; as we have defined them a shallow-to-deep transition increments  $t$  but a deep-to-shallow transition does not. A shallow jog has two transitions to its reflection, incrementing  $t$ . Finally,  $N(0) = 1$  since there is exactly one way to tile a  $4 \times 0$  rectangle.

This gives us a  $3 \times 3$  transfer matrix, and the system of linear equations

$$G = 1 + 4zG + 2zG_1$$

$$G_1 = zG + 2zG_2$$

$$G_2 = 2G_1 + 2zG_2$$

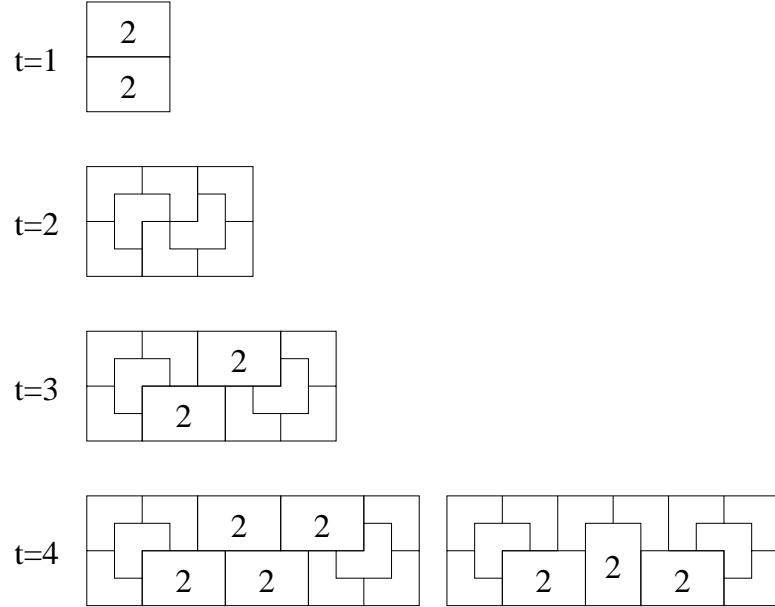
The solution for  $G$  is

$$G(z) = \frac{1 - 6z}{1 - 10z + 22z^2 + 4z^3}$$

The coefficients  $N(t)$  of the first few terms of  $G$ 's Taylor expansion are

$$1, 4, 18, 88, 468, 2672, 16072, 100064, 636368, 4097984, 26579488, \dots$$

giving the number of ways to tile rectangles of size  $4 \times 0$ ,  $4 \times 3$ ,  $4 \times 6$ , and so on.



**Fig. 2.** Fault-free rectangles of right trominoes of width 4 and length 3, 6, 9, and 12.

To gain a better understanding of this series, we concentrate on *fault-free* rectangles, those whose only straight interfaces are those at the left and right end. The first few of these are shown in Figure 2. These always begin and end with transitions from a straight edge to a deep jog and back, except for  $t = 1$  where we have a pair of  $2 \times 3$  rectangles. By disallowing transitions back to a straight interface except at the end, we find that the linear equations for fault-free rectangles are

$$G' = 4z + 2zG'_1$$

$$G'_1 = z + 2zG'_2$$

$$G'_2 = 2G'_1 + 2zG'_2$$

This has the solution

$$G'(z) = 2z \frac{2 - 11z - 2z^2}{1 - 6z}$$

The Taylor series of  $G'$  tells us that the number  $N'$  of fault-free tilings is

$$4, 2, 8, 48, 288, 1728, \dots$$

which we can write as

$$N'(t) = \begin{cases} 4 & \text{if } t = 1 \\ 2 & \text{if } t = 2 \\ 8 \cdot 6^{t-3} & \text{if } t \geq 3 \end{cases}$$

Since any tiling consists of a concatenation of fault-free tilings, we have

$$G(z) = 1 + G'(z) + G'(z)^2 + G'(z)^3 + \dots = \frac{1}{1 - G'(z)}$$

which the reader can verify. Note that we have to define  $G'(0) = N'(0) = 0$  in order for this formula to work.

The asymptotic growth of  $N(t)$  is the reciprocal of the radius of convergence of  $G$ 's Taylor series. Thus  $N(t) \propto \lambda^t$  where  $\lambda$  is the largest positive root of

$$\lambda^3 - 10\lambda^2 + 22\lambda + 4 = 0$$

This gives

$$\lambda = \frac{2}{3} \left( 5 + \sqrt{34} \cos \frac{\theta}{3} \right)$$

where  $\pi/2 < \theta < \pi$  and

$$\tan \theta = -\frac{3}{11} \sqrt{\frac{519}{2}}$$

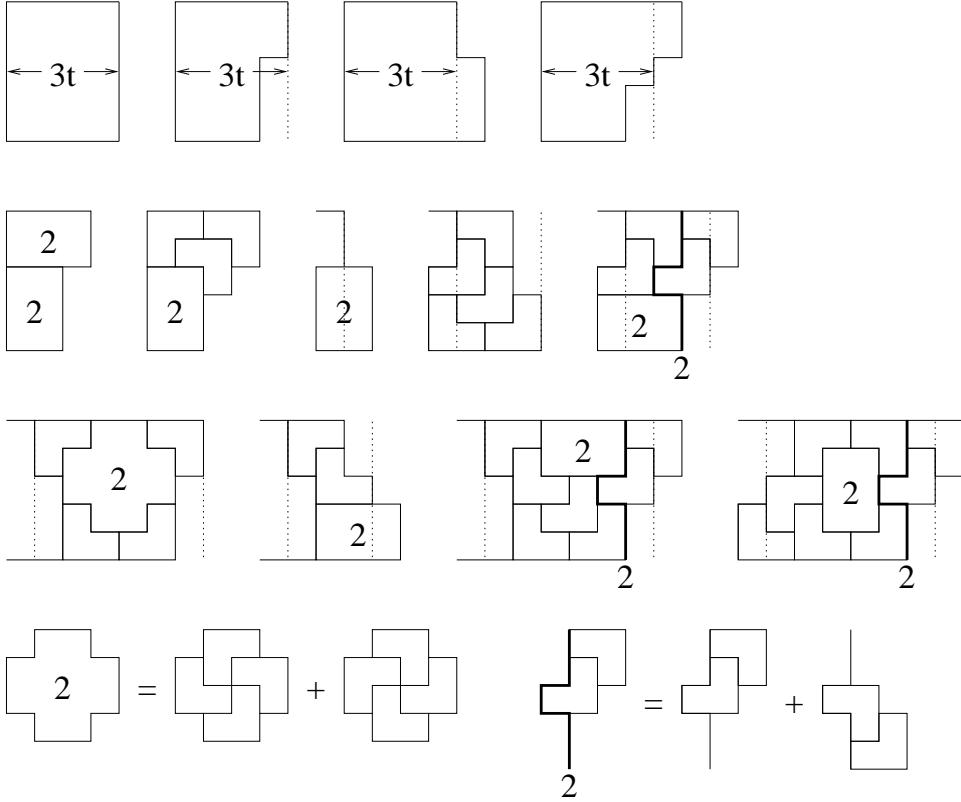
Numerically, we have

$$\lambda = 6.54560770847481152029 \dots$$

It would be nice to find a simpler expression for  $\lambda$ , perhaps using the decomposition into fault-free rectangles.

For  $5 \times 3t$  rectangles, four interface shapes suffice. We call these straight, little jog, big jog, and slope, and their transitions are shown in Figure 3 (reflections and reversals are not shown). If their generating functions are  $G$ ,  $G_1$ ,  $G_2$  and  $G_3$  respectively, our  $4 \times 4$  transfer matrix is

$$\begin{aligned} G &= 1 + 8zG_2 + 4zG_3 \\ G_1 &= 4zG + zG_1 + 2z^2G_2 + (2z + 4z^2)G_3 \\ G_2 &= 2G_1 + zG_2 + 4zG_3 \\ G_3 &= 2zG + 4zG_1 + (2z + 4z^2)G_2 + 4z^2G_3 \end{aligned}$$



**Fig. 3.** The four kinds of interfaces for the right tromino on rectangles of width 5, straight, big jog, little jog, and slope, and the transitions between them.

whose solution is

$$G(z) = \frac{1 - 2z - 31z^2 - 40z^3 - 20z^4}{1 - 2z - 103z^2 + 280z^3 + 380z^4}$$

The coefficients  $N(t)$  of the first few terms of  $G$ 's Taylor expansion are

$$1, 0, 72, 384, 8544, 76800, 1168512, 12785664, 170678784, 2014648320, 25633231872, \dots$$

Note that there are no  $5 \times 3$  tilings.

The generating functions for fault-free tilings obey

$$\begin{aligned} G' &= 8zG'_2 + 4zG'_3 \\ G'_1 &= 4z + zG'_1 + 2z^2G'_2 + (2z + 4z^2)G'_3 \\ G'_2 &= 2G'_1 + zG'_2 + 4zG'_3 \\ G'_3 &= 2z + 4zG'_1 + (2z + 4z^2)G'_2 + 4z^2G'_3 \end{aligned}$$

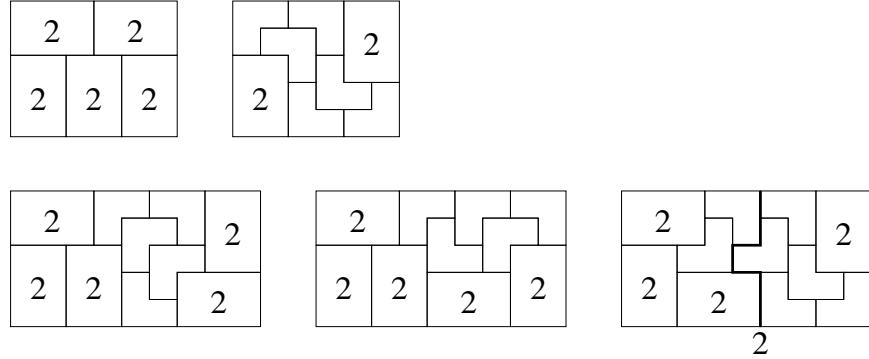
and so

$$G'(z) = 24z^2 \frac{3 + 10z + 15z^2}{1 - 2z - 31z^2 - 40z^3 - 20z^4}$$

The first few  $N'(t)$ , starting with  $t = 2$ , are then

$$72, 384, 3360, 21504, 163968, 1136640, 8283648, 58791936, 423121920, \dots$$

These are shown for  $t = 2$  and  $t = 3$  in Figure 4.



**Fig. 4.** Fault-free rectangles of right trominoes of width 5 and length 6 and 9.

The asymptotic growth of the number of fault-free rectangles is  $N'(t) \propto \lambda'^t$  where  $\lambda'$  is the largest positive root of

$$\lambda'^4 - 2\lambda'^3 - 31\lambda'^2 - 40\lambda' - 20 = 0$$

which has a rather complicated closed-form expression which we will not reproduce here. Numerically,

$$\lambda' = 7.16235536278185348653\dots$$

For all rectangles, including faulty ones, we have  $N(t) \propto \lambda^t$  where  $\lambda$  is the largest positive root of

$$\lambda^4 - 2\lambda^3 - 103\lambda^2 - 280\lambda - 380 = 0$$

This is

$$\lambda = \frac{1}{2} + \frac{\sqrt{627y^{\frac{1}{3}} - 3y^{\frac{2}{3}} - 13107}}{6y^{\frac{1}{6}}} + \frac{\sqrt{4369 + 418y^{\frac{1}{3}} + y^{\frac{2}{3}} + 2304\sqrt{\frac{3y}{209y^{\frac{1}{3}} - y^{\frac{2}{3}} - 4369}}}}{2\sqrt{3}y^{\frac{1}{6}}}$$

where

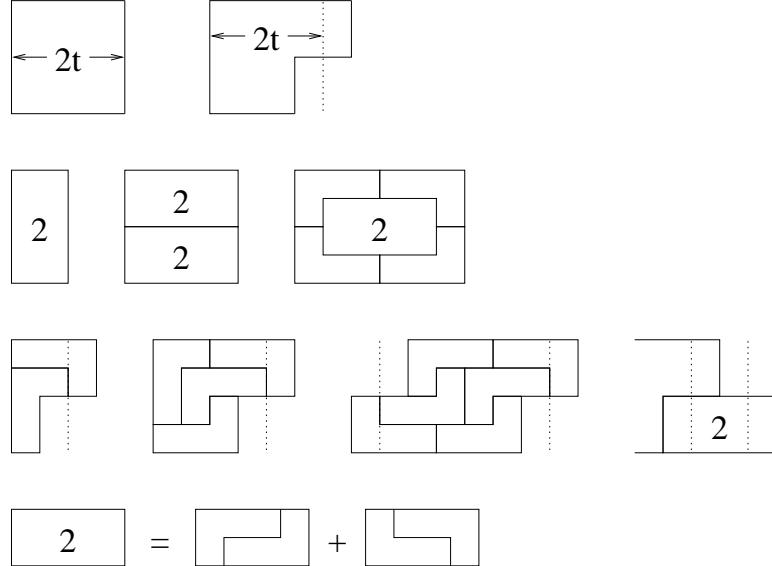
$$y = 1204327 - 72\sqrt{263697405}$$

Numerically, we have

$$\lambda = 12.36366722455963019234\dots$$

### 3 *L* tetrominoes

We can perform a similar analysis for tilings of  $4 \times n$  rectangles with *L* tetrominoes. In fact, here the analysis is easier. As Figure 5 shows, we have just two kinds of interfaces, straight and jogged. We define  $N(t)$  and  $N_1(t)$  as the number of tilings for a straight or jogged interface respectively when there are  $2t$  columns to the left of the dotted line. Then a straight has two transitions to itself that increase  $t$  by 1, four to itself that increase  $t$  by 2, and two to itself that increase  $t$  by 3. It has two transitions each to a jog and its reflection, one of which increases  $t$  by 1 and the other by 2. Finally, a jog has two transitions to another (reflected) jog that increase  $t$  by 1, one transition to itself that increases  $t$  by 3, and the two inverse transitions back to a straight.



**Fig. 5.** The two kinds of interfaces for the *L* tetromino on rectangles of width 4, straight and jogged, and the transitions between them.

Thus we have a  $2 \times 2$  transfer matrix, and defining generating functions as before gives us the linear equations

$$\begin{aligned} G &= 1 + (2z + 4z^2 + 2z^3)G + (2z + 2z^2)G_1 \\ G_1 &= (z + z^2)G + (2z + z^3)G_1 \end{aligned}$$

so

$$G(z) = \frac{1 - 2z - z^3}{1 - 4z - 2z^2 + z^3 + 4z^4 + 4z^5 + 2z^6}$$

The coefficients  $N(t)$  of the first few terms of  $G$ 's Taylor expansion are

$$1, 2, 10, 42, 182, 790, 3432, 14914, 64814, 281680, 1224182, \dots$$

giving the number of ways to tile a rectangle of size  $4 \times 0$ ,  $4 \times 2$ ,  $4 \times 4$ , etc.

As before, we can focus our attention on fault-free rectangles. For  $t > 3$ , any such rectangle consists of transitions from a straight edge to a jog, and transitions between jogs in between. For  $t = 1, 2, 3$  we have an additional 2, 4, and 2 fault-free rectangles respectively as shown in Figure 5. The linear equations for the fault-free generating functions are thus

$$\begin{aligned} G' &= 2z + 4z^2 + 2z^3 + 2(z + z^2)G'_1 \\ G'_1 &= z + z^2 + (2z + z^3)G'_1 \end{aligned}$$

This gives

$$G'(z) = \frac{2z(1+z)^2(1-z-z^3)}{1-2z-z^3}$$

whose Taylor expansion tells us that the number  $N'$  of fault-free tilings is

$$2, 6, 10, 18, 38, 84, 186, 410, 904, 1994 \dots$$

The first few of these are shown in Figure 6. The reader can verify that  $G(z) = 1/(1 - G'(z))$ .

The asymptotic growth of the number of fault-free rectangles is  $N'(t) \propto \lambda'^t$  where  $\lambda'$  is the largest root of

$$\lambda'^3 - 2\lambda'^2 - 1 = 0$$

which is

$$\lambda' = \frac{1}{3} \left( 2 + \sqrt[3]{\frac{43}{2} - \frac{3\sqrt{177}}{2}} + \sqrt[3]{\frac{43}{2} + \frac{3\sqrt{177}}{2}} \right)$$

or numerically

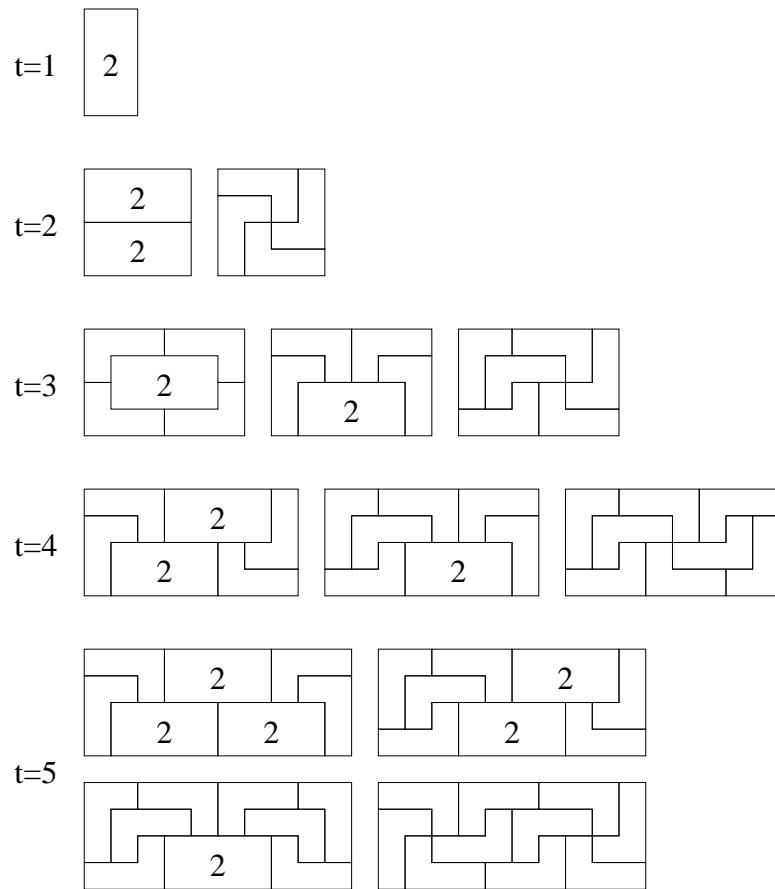
$$\lambda' = 2.20556943040059031170 \dots$$

For all rectangles, including faulty ones, we have  $N(t) \propto \lambda^t$  where  $\lambda$  is the largest root of

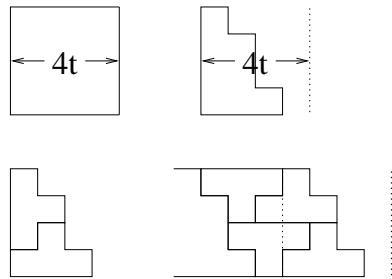
$$\lambda^6 - 4\lambda^5 - 2\lambda^4 + \lambda^3 + 4\lambda^2 + 4\lambda + 2 = 0$$

This appears not to have a closed form, but it is approximately

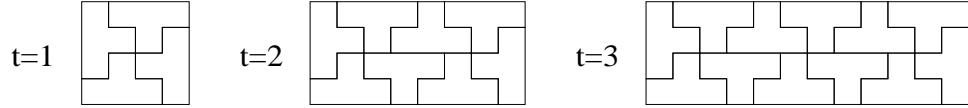
$$\lambda = 4.34601641141649282849 \dots$$



**Fig. 6.** Fault-free rectangles of  $L$  tetrominoes of width 4 and length 2, 4, 6, 8, and 10.



**Fig. 7.** The two kinds of interfaces for the  $T$ -tetromino, straight and jagged, and the transitions between them.



**Fig. 8.** The  $T$  tetromino has two fault-free rectangles of each size.

#### 4 $T$ tetrominoes

Of these three polyominoes, finding the number of tilings of a  $4 \times n$  rectangle is easiest for the  $T$  tetromino. In fact, such tilings only exist if  $n$  is a multiple of 4. Figure 7 shows two kinds of interfaces, straight and jagged, with  $4t$  columns to the left of the dotted line. A straight edge must make a transition either to a jag or its reflection, incrementing  $t$ . A jag can make a transition back to straight, leaving  $t$  fixed, or to itself, incrementing  $t$ . Thus the generating functions obey

$$G = 1 + 2G_1$$

$$G_1 = zG + zG_1$$

This gives

$$G(z) = \frac{1-z}{1-3z}$$

whose Taylor expansion tells us that

$$N(t) = \begin{cases} 1 & \text{if } t = 0 \\ 2 \cdot 3^{t-1} & \text{if } t > 0 \end{cases}$$

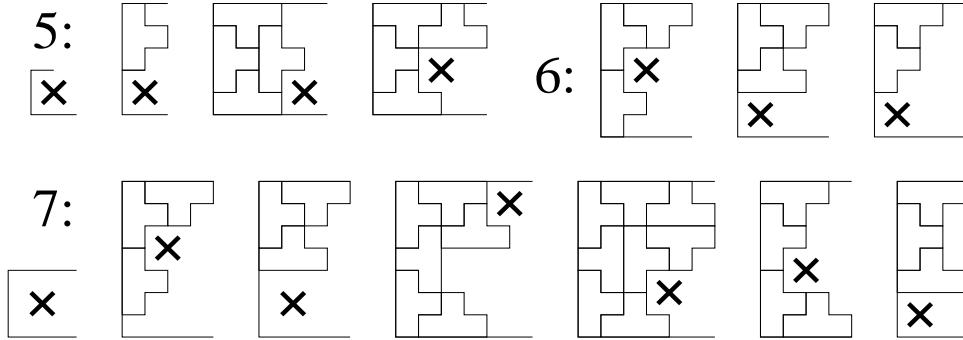
Thus we can find a closed form for  $N(t)$ , unlike in the previous two cases. There are exactly two fault-free rectangles for each  $t$ , as shown in Figure 8. Thus the number of tilings is  $2 \cdot 3^{t-1}$  since there are two choices of initial fault-free rectangle and three choices for each increment of  $t$ , namely either continuing the current fault-free rectangle, or ending it and starting one of two new ones.

It is easy to show that the  $T$  tetromino cannot tile any rectangles of width 5, 6, or 7. Proofs are shown in Figure 9. More generally, Walkup [6] showed that a rectangle can be tiled with  $T$  tetrominoes if and only if its length and width are both multiples of 4.

#### 5 Tilings of the plane

Since the plane can be divided into rows or columns, the entropy per site of tilings of rectangles of a given width serves as a lower bound for the entropy per site of tilings of the plane. Specifically, if the number of ways to tile a  $m \times n$  rectangle is  $N(m, n)$ , we define the entropy  $\sigma$  as

$$\sigma = \lim_{m, n \rightarrow \infty} \frac{\ln N(m, n)}{mn}$$



**Fig. 9.** Proofs that the  $T$  tetromino cannot tile rectangles of width 5, 6, or 7. Attempted tilings always lead to one of the un-tileable notches shown on the left, or to an even more obviously un-tileable space.

so that

$$N(m, n) \propto e^{\sigma mn}$$

for large rectangles. For the right tromino, our analysis of  $5 \times n$  rectangles gives

$$\sigma \geq \frac{1}{15} \ln \lambda = 0.1676508 \dots$$

where the factor of 15 comes from the fact that each increment of  $t$  adds 15 sites to the rectangle. Similarly, for the  $L$  tetromino on  $4 \times n$  rectangles, we have

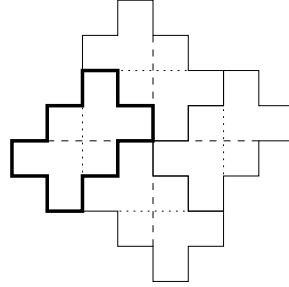
$$\sigma \geq \frac{1}{8} \ln \lambda = 0.183657 \dots$$

Better lower bounds could be obtained by looking at wider rectangles. Note that if there are any tilings we failed to see, this only improves these lower bounds, since additional transitions can only increase  $\lambda$ .

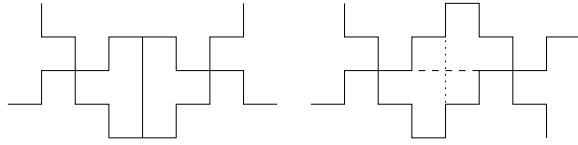
For the  $T$  tetromino, the lower bound we get from  $4 \times n$  rectangles,  $\frac{1}{16} \ln 3 \approx 0.06866$ , is not as good as the bound  $\frac{1}{8} \ln 2 \approx 0.08664$  which we can derive by noting that the 8-cell shape in Figure 10 can tile the plane, and in turn can be tiled in two ways by the  $T$ . We can get a better bound as follows.

Suppose that the boundaries between tiles at points  $(4x, 4y)$  are clockwise or counterclockwise fylfots as shown in Figure 11. Whenever neighboring fylfots have opposite orientations, there is only one way to tile the space between them, but if they have the same orientation the space between them can be tiled in two ways as in Figure 10. (Note that this is also the source of two of the choices in the  $4 \times n$  tilings above.) If we define the two orientations as  $+1$  and  $-1$ , then we can sum over all configurations of fylfots  $\{s_i = \pm 1\}$ , with each one giving us 1 choice for pairs of unlike neighbors and 2 choices for like ones. Thus we have a lower bound of

$$N(m, n) = \sum_{\{s_i = \pm 1\}} \prod_{ij} \left\{ \begin{array}{l} 1 \text{ if } s_i s_j = -1 \\ 2 \text{ if } s_i s_j = +1 \end{array} \right\}$$



**Fig. 10.** Since this eight-cell shape can tile the plane and in turn can be tiled in two ways, the entropy of tilings of the plane with  $T$  tetrominoes is at least  $\frac{1}{8} \ln 2$ .



**Fig. 11.** The space between two fylfots 4 sites apart can be tiled in 1 way if they are of opposite type, and 2 ways if they are the same.

where the product is over all pairs of nearest neighbors, and the sum is over the configurations of a lattice of size  $\frac{m}{4} \times \frac{n}{4} = \frac{mn}{16}$ . We can rewrite this as

$$\begin{aligned} N(m, n) &= \sum_{\{s_i = \pm 1\}} \prod_{ij} e^{\frac{1}{2}(s_i s_j + 1) \ln 2} \\ &= 2^{\frac{mn}{16}} \sum_{\{s_i = \pm 1\}} \prod_{ij} e^{(\frac{1}{2} \ln 2) s_i s_j} \end{aligned}$$

(note that there are  $\frac{mn}{8}$  edges in the fylfot lattice).

We now note that the latter sum is the partition function of an antiferromagnetic Ising model with  $\beta = \frac{1}{2} \ln 2$ . We can transform this to a ferromagnetic model by negating the  $s_i$  on one of the checkerboard sublattices. Using the exact solution of the Ising model in two dimensions [7], we then have

$$\sigma \geq \frac{1}{16} (\ln 2 + \sigma_{\text{Ising}})$$

where

$$\sigma_{\text{Ising}} = \ln 2 + \frac{1}{2} \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\omega_1 d\omega_2 \ln [\cosh^2 2\beta - (\sinh 2\beta)(\cos \omega_1 + \cos \omega_2)]$$

For  $\beta = \frac{1}{2} \ln 2$ , where  $\cosh^2 2\beta = \frac{25}{16}$  and  $\sinh 2\beta = \frac{3}{4}$ , we have

$$\sigma_{\text{Ising}} = 0.8270269567 \dots$$

and so

$$\sigma \geq 0.09501088358\cdots$$

For upper bounds, we can generalize an argument given in [2] as follows. If we construct a tiling of the plane with right trominoes by scanning from top to bottom and left to right, at each step the first unoccupied site can be given a tromino with only four different orientations. Since we have at most  $n/3$  such choices, the entropy is at most  $\frac{1}{3} \ln 4 \approx 0.462$ . Similar considerations give  $\frac{1}{4} \ln 8 \approx 0.520$  and  $\frac{1}{4} \ln 4 \approx 0.347$  for the  $L$  and  $T$  tetromino respectively. Obviously the gap between our upper and lower bounds is still quite large.

For dominoes or ‘dimers,’ the entropy is known to be  $G/\pi = 0.29156\cdots$  where  $G$  is Catalan’s constant [1, 2]. It is tempting to think that other polyominoes might have exact solutions; one related model which does is the covering of the triangular lattice with triangular trimers [8]. However, while the general problem of covering an arbitrary graph with dimers can be solved in polynomial time, covering it with triangles is NP-complete [9], so generalized versions of the tromino problem are probably hard.

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*Note added.* Meyerowitz [10] has calculated the generating function for the  $T$  tetromino in rectangles of width 8.

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